MATH2060A Assignment 1

Deadline: Jan 18, 2019.

Hand in: Supplementary exercise no (1), (2bc), and (5).

Section 6.1 no 4, 7, 8cd, 9, 13, and 14.

Supplementary Exercises

1. Consider the function f defined on $[0,\infty)$

$$f(x) = x^\alpha \sin\frac{1}{x}\ , \quad \alpha > 0\ ,$$

and $f(0)=0$. Determine the range of α in which

   (a) f is continuous on $[0,\infty)$,

   (b) f is differentiable on $[0,\infty)$, and

   (c) f' exists and is differentiable on $[0,\infty)$.

2. Find (a) the maximal domain on which the function is well-defined, (b) the domain on which it is continuous and (c) the domain on which it is differentiable in each of the following cases. Justify your answer in (c).

   (a) $f(x) = |x^2-5x+6|$.

   (b) $h(x) = \log(16-x^2)$.

   (c) $j(x) = \cos|x|$.

3. Find a function which is not differentiable exactly at the following points on $(-\infty,\infty)$ in each of the following cases:

   (a) n-many distinct points $\{a_1, a_2, \cdots, a_n\}$,

   (b) The set of integers $\mathbb{Z}$, and

   (c) $\left\{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots, \right\}$.

4. A function $f:(a,b)\to\mathbb{R}$ has a symmetric derivative at $c\in(a,b)$ if

$$f_s'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h}$$

exists. Show that $f_s'(c) = f'(c)$ if the latter exists. But $f_s'(c)$ may exist even though f is not differentiable at c. Can you give an example?

5. Let $f:\mathbb{R}\to\mathbb{R}$ satisfy $f(x+y)=f(x)f(y)$ for all $x,y\in\mathbb{R}$. Suppose f is differentiable at 0 with $f'(0)=1$. Show that f is differentiable on $\mathbb{R}$ and $f'(x)=f(x)$ for all $x\in\mathbb{R}$.

Continuity and Differentiability

Proposition. *Let f be defined on (a,b) and $x_0 \in (a,b)$. If f is differentiable at x_0, it is also continuous at x_0.* Proof: As f is differentiable at x_0, for $\varepsilon = 1$, there is some δ_1 such that

$$\left| \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) \right| < 1, \quad x \in (x_0-\delta_1, x_0+\delta_1) \setminus \{x_0\} \ .$$

It follows that

$$|f(x)-f(x_0)-f'(x_0)(x-x_0)| \le |x-x_0| \ ,$$

for all $x \in (x_0-\delta_1, x_0+\delta_1)$. Therefore,

$$\begin{aligned} |f(x)-f(x_0)| &\le |f(x)-f(x_0)-f'(x_0)(x-x_0)| + |f'(x_0)(x-x_0)| \\ &\le (1+|f'(x_0)|)|x-x_0| \ , \quad \forall x \in (x_0-\delta_1, x_0+\delta_1) \ . \end{aligned}$$

Now, given $\varepsilon > 0$, we can find some $\delta \le \delta_1$ such that

$$\delta < \frac{\varepsilon}{1+|f'(x_0)|} \ .$$

Then,

$$|f(x)-f(x_0)| \le \varepsilon, \quad \forall x \in (x_0-\delta, x_0+\delta) \ ,$$

done.

In our text book, there is a short proof based on the Limit Theorem. Here we show how to use the $\varepsilon - \delta$ argument to the same effect.

There are plenty examples showing continuity does not imply differentiability. Some typical examples were discussed in class, including the following three:

- The function $f(x) = |x|$ is not differentiable at $x_0 = 0$. (Reason: The left derivative and right derivative at 0, although exist, do not match.)
- The function $g(x) = |x|^{1/2}$ is not differentiable at $x_0 = 0$. (Reason: The difference quotient tends to infinity as x goes to 0.)
- The function $h(x) = x \sin 1/x$ (and set $h(0) = 0$) is not differentiable at $x_0 = 0$. (Reason: The difference quotient does not converge due to rapid oscillation.)

Elementary Functions

Here we summarize the formulas of derivatives of elementary functions and outline how they are proved.

(1) **Polynomials**

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1},$$

where

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n \ , \forall x \in \mathbb{R}.$$

(2) **Rational functions**

$$\left(\frac{p(x)}{q(x)}\right)' = \frac{p'(x)q(x) - p(x)q'(x)}{q^2(x)} \ ,$$

where p, q are polynomials and $q(x) \neq 0$. It is defined on $\{x : \ q(x) \neq 0\}$ and differentiable there.

(3) **The radical**

$$\left(x^{1/n}\right)' = \frac{1}{n}x^{1/n-1} \ ,$$

where $x \geq 0$ and $n \geq 2$. It is defined and continuous on $[0, \infty)$ and differentiable on $(0, \infty)$.

(4) **The exponential and logarithmic functions**

$$(e^x)' = e^x, \ \forall x \in \mathbb{R}, \quad (\log x)' = \frac{1}{x} \ , \forall x > 0.$$

The exponential function is defined and differentiable on $(-\infty, \infty)$ and the logarithmic function is defined and differentiable on $(0, \infty)$.

(5) **The sine and cosine functions**

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x \ , \quad \forall x \in \mathbb{R}.$$

Both functions are differentiable everywhere.

(6) **The absolute value function**

$$(|x|)' = \text{sign } (x), \quad x \neq 0 \ .$$

It is continuous on $(-\infty, \infty)$ and differentiable on $(-\infty, 0) \cap (0, \infty)$. However, it is not differentiable at $x = 0$. Its right and left derivatives exist but are not equal.

Proof of (1) By linearity and the product rule.

Proof of (2) By the quotient rule and (1).

Proof of (3) By the inverse rule and chain rule. When $\alpha = 1/n, n \geq 1$, $x^{1/n}$ is the inverse function of x^n for $x > 0$. When $\alpha = p/q, p, q > 0$, recall that $x^\alpha = x^{p/q} = (x^{1/q})^p = (x^p)^{1/q}$. When $p < 0, q > 0$, recall that $x^\alpha = 1/(x^{1/q})^{-p}$. Later we will extend this formula to all non-zero real number α after the meaning of x^α for irrational α is given.

Proof of (4) You may take the first formula for granted. It will be proved after the "ultimate" definition of the exponential function is introduced in a later chapter. The second formula follows from the first one by the inverse rule.

Proof of (5) You may take this for granted or follow my formal proof in class. They will be proved once more after the "ultimate" definition of the sine and cosine functions is introduced in a later chapter.

Proof of (6) Use definition.

Derivatives of many functions can be computed using the formulas in (1) to (6) together with linearity, the product, quotient, chain and inverse rules.